Int. J. Heat Mass Transfer. Vol. 26, No. 11, pp. 1597–1601, 1983
Printed in Great Britain

0017-9310/83 $3.00 + 0.00
Pergamon Press Ltd.

THE EXACT SOLUTION OF THE NUSSELT'S MODEL OF THE CROSS-FLOW RECUPERATOR

JAN LACH
Department of Reactor Engineering, Institute of Nuclear Research, 05-400 Otwock-Świerk, Poland

(*Received* 25 *October* 1982)

Abstract—The exact solution of the Nusselt's model of the cross-flow recuperator obtained by the use of the Mikusinski operator calculus is presented in this work. The solution enables formulas for basic recuperator parameters to be derived. Thus it is no longer necessary to perform calculations using formulas for the counter-flow recuperator and the appropriate correction factors.

NOMENCLATURE

a	parameter, kF/W_1
b	parameter, kF/W_2
Bes_n	family of functions of two variables defined by double power series (10)
Bs_n	family of functions of two variables defined by double power series (17)
f	integrable function prescribed as a boundary condition
F	heat transfer surface
g	integrable function prescribed as a boundary condition
k	constant coefficient of heat transfer or an integer
n	integer number
$\dot{Q}_0$	thermal load of recuperator
s_1	operator $1/\{1\}$ with respect to the variable η
s_2	operator $1/\{1\}$ with respect to the variable ξ
T	medium temperature
W	medium heat capacity
x	dimensional coordinate
y	dimensional coordinate

Greek symbols

α	real number
η	dimensionless coordinate, y/y_0
ξ	dimensionless coordinate, x/x_0
Φ_k	efficiency of recuperator
Θ	dimensionless temperature, $(T-T'_2)/(T'_1-T'_2)$
μ, ρ, ω	integration variables or parameters

Subscripts

1	value of heating medium parameters
2	value of cooling medium parameters
m	mean value of the media temperature difference
0	maximum value of the dimensional coordinate

Superscripts

$'$	inlet value of the medium temperature
$''$	outlet value of the medium temperature

1. INTRODUCTION

AN ANALYSIS of the two-media cross-flow recuperator is presented. The theory of this recuperator type was initiated by Nusselt [1, 2]. In Nusselt's approach, the mathematical model consists of a system of first-order linear partial differential equations with constant coefficients. This system of equations is usually transformed into a Volterra integral equation of the second kind with difference kernel. It may be solved by Picard's method of successive approximations. Nusselt's method is often adduced in classical monographs on heat transfer theory [3].

In the present paper, a different method of solving the recuperator balance energy equations is demonstrated. The system of differential equations is not transformed into a Volterra integral equation. It is instead solved immediately using Mikusinski's operational calculus [4]. In this way the exact solution of the problem is obtained. It is shown that this solution can be expressed in the form of a convolution of analytic functions with known functions as boundary conditions.

Such a form of solution enables formulas for the effective calculation of basic recuperator parameters, such as thermal load, efficiency, mean temperature difference and total surface, to be found.

2. FORMULATION OF THE PROBLEM

Formulating the mathematical model of two-media cross-flow recuperator, we assume the following simplifications:

(1) The heat transfer surface is considered to be a rectangular plate, $F = x_0 y_0$.

(2) Heat transfer between the heating and cooling media takes place only in the transverse direction and heat conduction along the plate is negligible.

(3) Heat capacities of the media W_1 and W_2 and the heat transfer coefficient are constant along the flowing paths of the media.

(4) Heating and cooling media are flowing across the channel cross-sections x_0 and y_0 at uniformly distributed velocities.

(5) The gradient of the temperature exists only in the direction transverse to the media flow paths.

The model considered is shown by Fig. 1. The assumed simplifications are identical to the assumptions made in a similar case by Nusselt [1, 2].

Taking into account these simplifications and introducing the dimensionless variables defined in the nomenclature, we obtain the mathematical model of the considered recuperator in the form of the following set of differential equations:

$$\frac{\partial \Theta_1}{\partial \xi} = -a\Theta_1 + a\Theta_2, \qquad \frac{\partial \Theta_2}{\partial \eta} = b\Theta_1 - b\Theta_2. \tag{1}$$

The boundary conditions associated with this set of equations are as follows:

$$\Theta_1(\xi = 0, \eta) = 1, \qquad \Theta_2(\xi, \eta = 0) = 0. \tag{2}$$

3. ANALYTIC SOLUTION BY THE MIKUSINSKI OPERATOR CALCULUS

The above formulated problem may be solved using Mikusinski operator calculus [4].

Assume that the boundary conditions (2) are of the form

$$\Theta_1(\xi = 0, \eta) = f(\eta), \qquad \Theta_2(\xi, \eta = 0) = g(\xi). \tag{3}$$

In order to determine the temperature distribution of the heating medium, let us write the second equation of the system (1) in the operator form

$$s_1\{\Theta_2(\xi,\eta)\} - g(\xi) = b\{\Theta_1(\xi,\eta)\} - b\{\Theta_2(\xi,\eta)\} \tag{4}$$

thus obtaining

$$\{\Theta_2(\xi,\eta)\} = \frac{b}{s_1+b}\{\Theta_1(\xi,\eta)\} + \frac{1}{s_1+b}g(\xi). \tag{5}$$

Taking into account equation (5), we can rewrite the first equation of the system (1) in operator form as

$$\frac{\mathrm{d}\{\Theta_1(\xi,\eta)\}}{\mathrm{d}\xi} = \left(-a + \frac{ab}{s_1+b}\right)\{\Theta_1(\xi,\eta)\} + \frac{a}{s_1+b}g(\xi) \tag{6}$$

arriving at the solution [5]

$$\{\Theta_1(\xi,\eta)\} = \mathrm{e}^{[-a+(ab/s_1+b)]\xi}\{f(\eta)\} + a\int_0^{\xi} \frac{1}{s_1+b} g(\omega)\,\mathrm{e}^{[-a+(ab/s_1+b)](\xi-\omega)}\,\mathrm{d}\omega. \tag{7}$$

The formula (7) can be rewritten in the form

$$\Theta_1(\xi,\eta) = \mathrm{e}^{-a\xi}f(\eta) + \int_0^{\eta} f(\mu)\,\mathrm{e}^{-[a\xi+b(\eta-\mu)]} \times Bes_1(ab\xi, \eta-\mu)\,\mathrm{d}\mu + a\int_0^{\xi} g(\omega)\,\mathrm{e}^{-[a(\xi-\omega)+b\eta]} \times Bes_0(ab(\xi-\omega),\eta)\,\mathrm{d}\omega \tag{8}$$

in view of the relations

$$\frac{1}{s_1+b}\mathrm{e}^{[-a+(ab/s_1+b)](\xi-\omega)} = \{\mathrm{e}^{-[a(\xi-\omega)+b\eta]}Bes_0(ab(\xi-\omega),\eta)\},$$
$$\mathrm{e}^{[-a+(ab/s_1+b)]\xi} - \mathrm{e}^{-a\xi} = \{\mathrm{e}^{-(a\xi+b\eta)}Bes_1(ab\xi,\eta)\} \tag{9}$$

where [6]

$$Bes_n(\xi,\eta) \stackrel{\mathrm{df}}{=} \sum_{k=\max(-n,0)}^{\infty} \frac{\xi^{k+n}\eta^k}{(k+n)!\,k!} \tag{10}$$

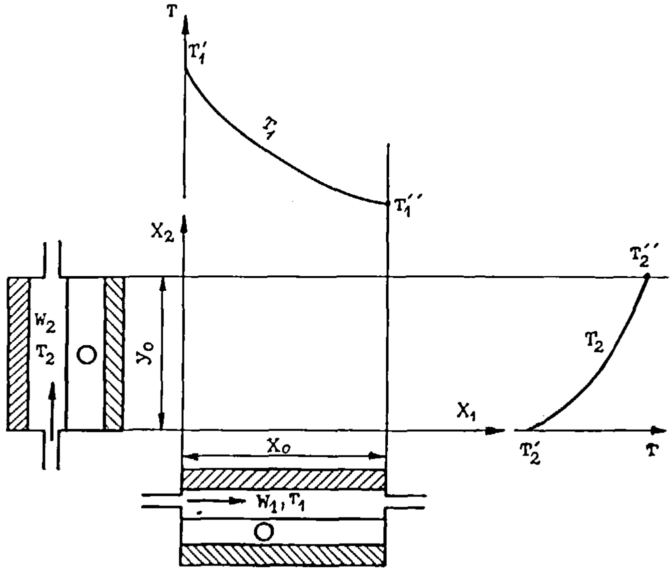

FIG. 1. The model of the two media cross-flow recuperator [3].

and of the definition of a convolution

$$e^{-(a\xi+b\eta)}Bes_1(ab\xi,\eta)*f(\eta)$$

$$=\int_0^\eta e^{-[a\xi+b(\eta-\mu)]}Bes_1(ab\xi,\eta-\mu)f(\mu)\,d\mu. \quad (11)$$

The temperature distribution of the cooling medium can be found in an analogous way, i.e. by rewriting the first equation of the system (1) in an operator form and determining the value of $\{\Theta_1(\xi,\eta)\}$. Next, substituting the value of $\{\Theta_1(\xi,\eta)\}$ into the second equation of the system (1) we obtain an operator differential equation

$$\frac{d\{\Theta_2(\xi,\eta)\}}{d\eta}=\left(\frac{ab}{s_2+a}-b\right)\{\Theta_2(\xi,\eta)\}+\frac{b}{s_2+a}f(\eta). \quad (12)$$

Making use of the relations (9) and the formula

$$e^{-(a\xi+b\eta)}Bes_1(ab\eta,\xi)*g(\xi)$$

$$=\int_0^\xi g(\rho)\,e^{-[a(\xi-\rho)+b\eta]}Bes_1(ab\eta,\xi-\rho)\,d\rho \quad (13)$$

we get the temperature distribution of the cooling medium in the form

$$\Theta_2(\xi,\eta)=e^{-b\eta}g(\xi)+\int_0^\xi g(\rho)\,e^{-[a(\xi-\rho)+b\eta]}$$

$$\times\, Bes_1(ab\eta,\xi-\rho)\,d\rho+b\int_0^\eta f(\rho)\,e^{-[a\xi+b(\eta-\rho)]}$$

$$\times\, Bes_0(ab(\eta-\rho),\xi)\,d\rho. \quad (14)$$

The solutions (8) and (14) are general in the sense that they remain valid for many various boundary conditions. The problem considered has been reduced to the calculation of integrals of functions which are products of analytic functions and of some known functions occurring in the boundary conditions (3).

Now assume that boundary conditions are of the form (2), that is

$$f(\eta)=1,\qquad g(\xi)=0. \quad (15)$$

In this case, we arrive at the formulas

$$\Theta_1(\xi,\eta)=e^{-a\xi}+e^{-(a\xi+b\eta)}\int_0^\eta e^{b\mu}\,Bes_1(ab\xi,\eta-\mu)\,d\mu,$$

$$\Theta_2(\xi,\eta)=b\,e^{-(a\xi+b\eta)}\int_0^\eta e^{b\rho}\,Bes_0(ab(\eta-\rho),\varepsilon)\,d\rho. \quad (16)$$

Defining a new family of functions [6]

$$Bs_n(\xi,\eta)=\sum_{m=\max(0,n)}^{\infty}\frac{\xi^m}{m!}\sum_{k=0}^{m-n}\frac{\eta^k}{k!} \quad (17)$$

and taking into account the formula

$$\int e^{\alpha\eta}\,Bes_n(\xi,\eta)\,d\eta=$$

$$-(-\alpha)^{n-1}\,e^{\alpha\eta}Bs_n\left(-\frac{\xi}{\alpha},-\alpha\eta\right),\quad \alpha\neq 0 \quad (18)$$

one can calculate the integrals occurring in (16). In this way we get a final form of the solution of the problem (1, 2)

$$\Theta_1(\xi,\eta)=1-e^{-(a\xi+b\eta)}Bs_1(a\xi,b\eta),$$

$$\Theta_2(\xi,\eta)=1-e^{-(a\xi+b\eta)}Bs_0(a\xi,b\eta). \quad (19)$$

Making use of the formulas

$$\frac{\partial Bs_n(\xi,\eta)}{\partial\xi}=Bs_{n-1}(\xi,\eta),$$

$$\frac{\partial Bs_n(\xi,\eta)}{\partial\eta}=Bs_{n+1}(\xi,\eta) \quad (20)$$

one can verify that the solutions (19) satisfy both the system differential equations (1) and the boundary conditions (2).

4. BASIC RECUPERATOR CHARACTERISTIC

Using the solution obtained in Section 3 basic recuperator parameters can now be determined.

It is relatively easy to prove that the thermal load of the recuperator is given by

$$\dot{Q}_0=W_1(T_1'-T_2')\int_0^1[1-\Theta_1(1,\eta)]\,d\eta. \quad (21)$$

Taking into account (19) and using the formula

$$\int e^{-\eta}Bs_1(\xi,\eta)\,d\eta=e^{-\eta}[\eta Bs_1(\xi,\eta)-\xi Bs_{-1}(\xi,\eta)] \quad (22)$$

gives

$$\dot{Q}_0=W_1(T_1'-T_2')\times\left\{\frac{a}{b}+e^{-(a+b)}\left[Bs_1(a,b)-\frac{a}{b}Bs_{-1}(a,b)\right]\right\}. \quad (23)$$

It is known, however, that there is relation between the thermal load and mean temperature difference

$$\dot{Q}_0=k(T_1-T_2)_mF. \quad (24)$$

By comparing equation (23) with equation (24) one obtains

$$\frac{(T_1-T_2)_m}{T_1'-T_2'}=\frac{1}{a}\left\{\frac{a}{b}+e^{-(a+b)}\left[Bs_1(a,b)-\frac{a}{b}Bs_{-1}(a,b)\right]\right\}. \quad (25)$$

From the tables of functions $Bs_n(\xi,\eta)$ it is possible to calculate the mean temperature difference and the thermal load of the recuperator or its total surface if the thermal load is given.

It is also possible to obtain expressions for calculating of media mean temperatures in the recuperators' outlets:

$$T_1'' = T_1' - (T_1' - T_2') \times \left\{ \frac{a}{b} + e^{-(a+b)} \left[Bs_1(a,b) - \frac{a}{b} Bs_{-1}(a,b) \right] \right\}, \quad (26)$$

$$T_2'' = T_2' + \frac{b}{a}(T_1' - T_2') \times \left\{ \frac{a}{b} + e^{-(a+b)} \left[Bs_1(a,b) - \frac{a}{b} Bs_{-1}(a,b) \right] \right\}.$$

If one replaces parameters a and b by the set of those of P and R,

$$P = \frac{T_2'' - T_2'}{T_1' - T_2'} = 1 + e^{-(a+b)} \left[\frac{b}{a} Bs_1(a,b) - Bs_{-1}(a,b) \right],$$

$$R = \frac{T_1' - T_1''}{T_2'' - T_2'} = \frac{a}{b} \quad (27)$$

then

$$\frac{(T_1 - T_2)_m}{T_1' - T_2'} = \frac{1}{a} PR,$$

$$\dot{Q}_0 = W_1(T_1' - T_2') PR. \quad (28)$$

The efficiency of the recuperator is given by

$$\Phi_k = \frac{T_1' - T_1''}{T_1' - T_2'} = PR$$

$$= \frac{a}{b} + e^{-(a+b)} \left[Bs_1(a,b) - \frac{a}{b} Bs_{-1}(a,b) \right]. \quad (29)$$

If the parameter P is replaced by

$$S = \frac{kF}{W_1} = a \quad (30)$$

then the efficiency Φ_k is expressed by

$$\Phi_k = R + e^{-(S+S/R)} \left[Bs_1\left(S, \frac{S}{R}\right) - R Bs_{-1}\left(S, \frac{S}{R}\right) \right]. \quad (31)$$

The relation Φ_k vs S with the parameter R is shown in Fig. 2.

Taking into account the following expressions:

$$Bs_n(0, \eta) = 0, \quad n > 0,$$
$$Bs_0(0, b) = Bs_{-1}(0, b) = e^b, \quad (32)$$
$$Bs_{-1}(S, 0) = e^S,$$

one can derive the following relations:

$$\lim_{S \to 0} \Phi_k = \lim_{a \to 0} \Phi_k = 0,$$
$$\left. \frac{\partial \Phi_k}{\partial S} \right|_{S=0} = \left. \frac{\partial \Phi_k}{\partial a} \right|_{a=0} = 1, \quad (33)$$
$$\lim_{R \to \infty} \Phi_k = 1 - e^{-S},$$

which do not depend on the recuperator design. Thus the relations given above must be also correct for the cross-flow recuperator under consideration.

5. CONCLUSIONS

The presented analysis method makes it possible to obtain the exact solution of the Nusselt's model of the cross-flow recuperator without the need to transform the set of balance equations (1) into a Volterra integral equation and apply the arduous method of the successive approximations.

Mikusinski's operational calculus, used for resolving the problem (1, 2), is more useful than Laplace transformations because it is easier to proceed from the operator form of the solution to its functional form.

The solution obtained, which is a convolution of analytic functions with those defined as the boundary

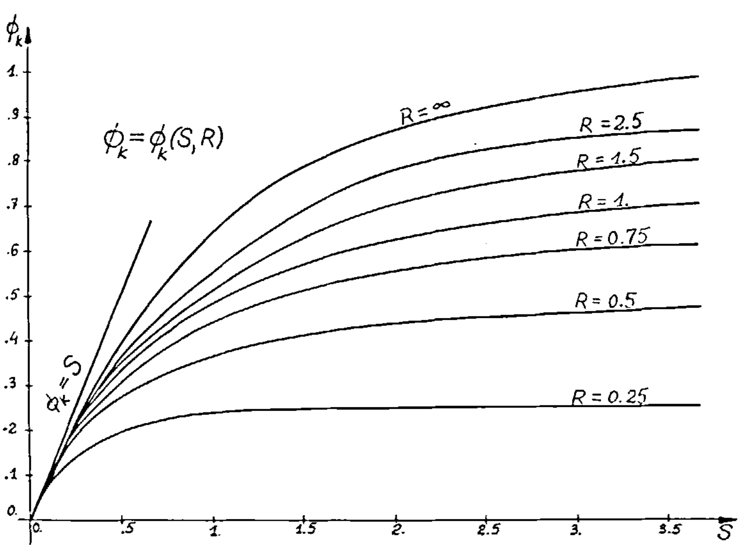

FIG. 2. The efficiency of the cross-flow recuperator as a function of the parameters S and R.

conditions, enables all functions f and g (3) to be taken into account.

The exact solution of the problem (1, 2) has made it possible to derive the formulas for basic cross-flow recuperator parameters. Thus it is not necessary to perform calculations using formulas for the counter-flow recuperator and the appropriate correction factors.

It should be noted that, making use of a real or complex transformation, one can reduce linear partial differential equations of the hyperbolic and elliptic type in the domain of two variables into a set of first-order partial differential equations with constant coefficients. The exact solution of such a set can then be obtained using the method presented in this paper.

REFERENCES

1. W. Nusselt, Der Wärmeübergang im Kreuzstrom, *Zeitschrift VDI* 2021 (1911).
2. W. Nusselt, Eine neue Formel für den Wärmedurchgang im Kreuzstrom, *Tech. Mech. Thermodyn.* **1**, 417 (1930).
3. J. Madejski, *Teoria wymiany ciepła* (*Theory of Heat Transfer*), pp. 357–363. PWN, Warszawa Poznań (1963).
4. J. Mikusiński, *Rachunek operatorów* (*Operational Calculus*). Polskie Towarzystwo Matematyczne (Polish Mathematical Society), Warszawa (1953).
5. E. Kamke, *Differentialgleichungen, Lösungsmethoden und Lösungen—I. Gewöhnliche Differentialgleichungen* (Russian transl.), p. 35. Izdat. Nauka, Moscou (1976).
6. W. Pieczka, On the general solution of some partial differential equation in two variables, *Demonstratio Mathematica* **7**, 451–462 (1974).

LA SOLUTION EXACTE DU MODÈLE DE NUSSELT DU RÉCUPÉRATEUR À COURANTS-CROISÉS

Résumé On présente la solution exacte du modèle de Nusselt du récupérateur à courants-croisés avec brassage qui est obtenue à l'aide du calcul opérationnel de Mikusinski. Cette solution permet à établir des formules lesquelles peuvent être appliquées directement dans un calcul des paramètres fondamentaux de ce type de récupérateur. Donc, le calcul comme s'il s'agissait d'un récupérateur à contre-courant et puis une utilisation des facteurs de correction n'est pas nécessaire.

EXAKTE LÖSUNG DES NUSSELTSCHEN PROBLEMS FÜR DEN KREUZSTROMWÄRMEAUSTAUSCHER

Zusammenfassung—In vorgelegtem Bericht ist die exakte Lösung für den Wärmedurchgang im Kreuzstromaustauscher angegeben. Dieses Problem, das zuerst von Nusselt formuliert wurde, ist mittels der Operatorenrechnung von Mikusiński gelöst worden. Die angegebene Lösung ermöglicht auch das Konstruieren einen entsprechenden Formel für die Berechnung der Grundparameter des Wärmeaustauschers. Man kann also allein mit Hilfe dieser Formel die notwendige Berechnung durchführen, ohne das bis jetzt gültige Lösungsverfahren zu benutzen.

АНАЛИТИЧЕСКОЕ РЕШЕНИЕ НУССЕЛЬТОВСКОЙ МОДЕЛИ ПЕРЕКРЕСТНОТОЧНОГО РЕКУПЕРАТОРА

Аннотация—В настоящей работе представлено аналитическое решение Нуссельтовской модели перекрестноточного рекуператора полученое путем применения исчисления операторов Микусинского. Такое решение позволяет на вывод формул предоставляющих возможность непосредственного расчета главных параметров этого рекуператора. Полученные формулы могут быть использованы непосредственно вместо расчетов как для рекуператора противоточного типа и применения соответствующих коэфициентов.